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1. Introduction

Let $\phi(z) \equiv \phi(x+iy)$ be a complex function which is integrable along a simple smooth path L in the complex z-plane. Let $\zeta = \xi + i\eta$ be a point on L which is not one of the end points of L and let ζ be imbedded in a subarc L_ε of L such that the end points of L_ε are equidistant from ζ. In this paper the symbol

$$\int_L \frac{\phi(z)dz}{z-\zeta}$$

means the Cauchy principal value, that is,

$$\int_L \frac{\phi(z)dz}{z-\zeta} = \lim_{\varepsilon \to 0} \int_{L-L_\varepsilon} \frac{\phi(z)dz}{z-\zeta}$$

provided the limit exists.

In a recent note [1] the author showed that the solution of the integral equation

$$(1.1) \qquad \int_0^1 \frac{\phi(x)dx}{x-\xi} = f(\xi) \; , \qquad 0 < \xi < 1$$

can be reduced to the solution of

$$(1.2) \qquad \int_0^x \frac{1}{\sqrt{x-t}} \int_t^1 \frac{\sqrt{\xi}\,\phi(\xi)d\xi dt}{\sqrt{\xi-t}} = \int_0^x \sqrt{\xi}\, f(\xi)d\xi + 2c\sqrt{x}$$

and therefore to the solution of the pair of Volterra equations

$$(1.3) \qquad \int_0^x \frac{\psi(t)dt}{\sqrt{x-t}} = \int_0^x \sqrt{\xi}\, f(\xi)d\xi + 2c\sqrt{x}$$

$$(1.4) \qquad \int_t^1 \frac{\sqrt{\xi}\,\phi(\xi)d\xi}{\sqrt{\xi - t}} = \psi(t) \ .$$

Equation (1.3) is Abel's equation and equation (1.4) can be reduced to the form (1.3) by using the transformation $\xi = 1 - u$; $\sigma = 1 - v$. The solution of (1.3) followed by the solution of (1.4) gives

$$(1.5) \quad \sqrt{\xi}\,\phi(\xi) = \frac{c}{\pi\sqrt{1-\xi}} - \frac{1}{\pi^2}\frac{d}{d\xi}\int_\xi^1 \frac{1}{\sqrt{t-\xi}}\int_0^t \frac{\sqrt{x}\, f(x)dxdt}{\sqrt{t-x}} \ .$$

The analysis in [1] also implies that the solution of (1.1) can be based on a knowledge of the integral

$$(1.6) \qquad -\int_0^t \frac{1}{\sqrt{\xi(t-x)}} \cdot \frac{d\xi}{\xi - x} = \begin{cases} 0 & 0 < x < t \\ \dfrac{\pi}{\sqrt{x(x-t)}} & t < x \end{cases}$$

and the solution of just one Abel equation in the following way.

In order to allow $\phi(x)$ to possess a possible singularity at $x = 0$ let (1.1) be written in the form

$$(1.7) \qquad \int_0^1 \frac{x\phi(x)dx}{x - \xi} = \xi f(\xi) + c$$

where

$$c = \int_0^1 \phi(x)dx \ .$$

The multiplication of both sides of (1.7) by $1/\sqrt{\xi(t-\xi)}$ followed by an integration gives

$$\int_0^t \frac{1}{\sqrt{\xi(t-\xi)}} \int_0^1 \frac{x\phi(x)dxd\xi}{x-\xi} = \int_0^t \frac{\sqrt{\xi}\ f(\xi)d\xi}{\sqrt{t-\xi}} + c\int_0^t \frac{d\xi}{\sqrt{\xi(t-\xi)}}$$

or

$$(1.8) \qquad -\int_0^1 x\phi(x)\int_0^t \frac{1}{(\xi - x)\sqrt{\xi(t-\xi)}}\, d\xi dx = \int_0^t \frac{\sqrt{x}\ f(x)dx}{\sqrt{t-x}} + \pi c \ .$$

From (1.6) the equation (1.8) is the same as

$$\int_t^1 \frac{\sqrt{x}\ \phi(x)dx}{\sqrt{x-t}} = c + \frac{1}{\pi}\int_0^t \frac{\sqrt{x}\ f(x)dx}{\sqrt{t-x}} \ .$$

The solution of this integral equation is readily found to be

$$\sqrt{x}\ \phi(x) = \frac{c}{\pi\sqrt{1-x}} - \frac{1}{\pi^2}\frac{d}{dx}\int_x^1 \frac{1}{\sqrt{t-x}}\int_0^t \frac{\sqrt{x}\ f(x)dxdt}{\sqrt{t-x}}$$

which agrees with (1.5).

The method presented in the last paragraph depends on (1.6) which shows that

$$\psi(\xi) = \frac{1}{\sqrt{\xi(t-\xi)}}$$

is a solution of the homogeneous equation

$$\int_0^t \frac{\psi(\xi)d\xi}{\xi - x} = 0 \ , \qquad 0 < x < \xi \ .$$

More generally, it may be possible to base the solution of the Fredholm equation

$$(1.9) \qquad \int_a^b K(\xi,x)\phi(x)dx = \lambda\phi(\xi) + f(\xi) \ , \qquad a < \xi < b$$

on the use of an explicit non-trivial solution $\psi(x,t)$ of

$$(1.10) \qquad \int_a^t K(x,\xi)\psi(x,t)dx = \lambda\psi(\xi,t) \quad , \qquad a < \xi < t \ .$$

If (1.9) is multiplied by $\psi(\xi,t)$ and then integrated it becomes

$$\int_a^t \psi(\xi,t) \int_a^b K(\xi,x)\phi(x)dxd\xi = \lambda \int_0^t \psi(\xi,t)\phi(\xi)d\xi + \int_0^t \psi(\xi,t)f(\xi)d\xi$$

or, if the order of integration can be changed,

$$\int_a^b \phi(x) \int_a^t K(\xi,x)\psi(\xi,t)d\_dx = \lambda \int_0^t \psi(\xi,t)\phi(\xi)d\xi + \int_0^t \psi(\xi,t)f(\xi)d\xi$$

which is the same as

$$\int_a^t \phi(x) \int_a^t K(\xi,x)\psi(\xi,t)d\xi dx + \int_t^b \phi(x) \int_a^t K(\xi,x)\psi(\xi,t)d\xi dx$$

$$= \lambda \int_a^t \psi(\xi,t)\phi(\xi)d\xi + \int_a^t \psi(\xi,t)f(\xi)d\xi \ .$$

From (1.10) this is equivalent to

$$(1.11) \qquad \int_t^b \phi(x) \int_a^t K(\xi,x)\psi(\xi,t)d\xi dx = \int_a^t \psi(\xi,t)f(\xi)d\xi \ .$$

If $K_1(t,x)$ denotes the value of the integral

$$\int_a^t K(\xi,x)\psi(\xi,t)d\xi$$

for $x > t$ the equation (1.11) becomes

$$(1.12) \qquad \int_t^b K_1(t,x)\phi(x)dx = \int_a^t \psi(\xi,t)f(\xi)d\xi \ .$$

Therefore if a non-trivial solution of (1.10) can be found for $a < t < b$ then it may be effective to reduce the solution of the Fredholm equation (1.9) in the above way to the solution of the Volterra equation (1.12).

When the kernel $K(\xi,x)$ is a Cauchy kernel, that is, when $K(\xi,x) = 1/(x - \xi)$ the method just presented can be used to extend the results of [1]. Section 2, below, shows that the above method can be used to reduce the solution of

$$(1.13) \qquad \int_0^1 \frac{\phi(x)dx}{x - \xi} = \lambda\phi(\xi) + f(\xi)$$

to the solution of Abel's integral equation. Section 3 demonstrates that the same ideas can be applied to obtain the solution of

$$(1.14) \qquad \int_L \frac{\phi(z)dz}{z - \zeta} = \lambda\phi(\zeta) + f(\zeta) \ .$$

In other words, the evaluation of a certain definite integral can be used to reduce the solution of (1.14) to the solution of a generalized Abel's equation. This leads to a new formula for the solution of (1.14).

2. Abel's Equation and a Cauchy Integral Equation of the Second Kind

The integral (1.6), namely

$$-\int_0^t \frac{d\xi}{(\xi - x)\sqrt{\xi(t - \xi)}}$$

is a special case of

$$-\int_0^t \left(\frac{t - \xi}{\xi}\right)^\gamma \frac{d\xi}{(t - \xi)(\xi - x)}$$

where $0 < \gamma < 1$. The substitution $t - \xi = \sigma\xi$ changes the last integral into

$$\int_0^\infty \frac{\sigma^{\gamma-1}dx}{x\sigma - (t-x)}\,d\sigma$$

which can be evaluated in a variety of ways. In one way or another it is not difficult to verify that

$$(2.1)\qquad -\int_0^t \left(\frac{t-\xi}{\xi}\right)^\gamma \frac{d\xi}{(t-\xi)(\xi-x)}$$

$$= \begin{cases} \frac{1}{x}\left(\frac{t-x}{x}\right)^{\gamma-1}\int_0^\infty \frac{\zeta^{\gamma-1}d\zeta}{\zeta-1} = -\frac{(t-x)^{\gamma-1}}{x^\gamma}\,\pi\cot\gamma\pi, & 0 < x < t \\ \frac{1}{x}\left(\frac{t-x}{x}\right)^{\gamma-1}\int_0^\infty \frac{\zeta^{\gamma-1}d\zeta}{\zeta+1} = \frac{(x-t)^{\gamma-1}}{x^\gamma}\cdot\frac{\pi}{\sin\gamma\pi}, & t < x\,. \end{cases}$$

This shows that if λ is a given real value and γ is chosen so that

$$(2.2)\qquad -\pi\cot\gamma\pi = \lambda\,, \qquad 0 < \gamma < 1$$

then

$$(2.3)\qquad \psi(\xi,t) = \frac{(t-\xi)^{\gamma-1}}{\xi^\gamma}$$

is a solution of

$$(2.4)\qquad -\int_0^t \frac{\psi(\xi,t)d\xi}{\xi-x} = \lambda\psi(x,t)\,, \qquad 0 < x < t\,.$$

Once this is recognized one can surmise that the solution of (2.4) can be used in various ways to deduce the solution of the nonhomogeneous equation

$$\int_0^1 \frac{\phi(x)dx}{x-\xi} = \lambda\phi(\xi) + f(\xi) \tag{2.5}$$

in which all quantities are supposed real. For example, (2.3) and (2.4) could be used in connection with the Hardy-Poincaré-Bertrand formula [2] to find the solution of (2.5) but the author believes that the simplest procedure is to use the method outlined in the introduction. In accordance with this method the solution (2.3) can be used to reduce (2.5) to a simple Volterra equation.

Equation (2.5) can be expressed in the form

$$\int_0^1 \frac{x\phi(x)dx}{x-\xi} = \lambda\xi\phi(\xi) + \xi f(\xi) + c \tag{2.6}$$

where

$$c = \int_0^1 \phi(x)dx .$$

Then, multiplication of (2.6) by $(t-\xi)^{\gamma-1}/\xi^{\gamma}$ and integration gives

$$\int_0^t \frac{(t-\xi)^{\gamma-1}}{\xi^{\gamma}} \int_0^1 \frac{x\phi(x)dxd\xi}{x-\xi} = \int_0^t \frac{(t-\xi)^{\gamma-1}}{\xi^{\gamma}}[\lambda\xi\phi(\xi) + \xi f(\xi) + c]d\xi . \tag{2.7}$$

Under the assumption that the order of integration can be changed, (2.7) is

$$(2.8)\qquad -\int_0^t x\phi(x)\int_0^t \frac{(t-\xi)^{\gamma-1}}{\xi^\gamma}\cdot\frac{d\xi dx}{\xi - x} - \int_t^1 x\phi(x)\int_0^t \frac{(t-\xi)^{\gamma-1}}{\xi^\gamma}\cdot\frac{d\xi dx}{\xi - x}$$

$$= \int_0^t \frac{(t-\xi)^{\gamma-1}}{\xi^\gamma}\,[\lambda\xi\phi(\xi) + \xi f(\xi) + c]d\xi\ .$$

With γ determined by (2.2), the evaluation (2.1) can be used to reduce (2.8) to

$$(2.9)\qquad \frac{\pi}{\sin\gamma\pi}\int_t^1 (x-t)^{\gamma-1}x^{1-\gamma}\phi(x)dx$$

$$= \int_0^t (t-\xi)^{\gamma-1}\xi^{1-\gamma}f(\xi)d\xi + c\int_0^t \frac{(t-\xi)^{\gamma-1}d\xi}{\xi^\gamma}\ .$$

Thus, since

$$\int_0^t \frac{(t-\xi)^{\gamma-1}d\xi}{\xi^\gamma} = \frac{\pi}{\sin\gamma\pi}\ ,$$

the solution of (2.5) reduces to the solution of

$$(2.10)\qquad \int_t^1 (x-t)^{\gamma-1}x^{1-\gamma}\phi(x)dx = \frac{\sin\gamma\pi}{\pi}\int_0^t (t-\xi)^{\gamma-1}\xi^{1-\gamma}f(\xi)d\xi + c\ .$$

Furthermore, it is not difficult to prove that the process can be reversed so that any solution of (2.10) is also a solution of (2.5).

The solution of Abel's integral equation

$$(2.11) \qquad \int_0^t \frac{\psi_1(x)dx}{(t-x)^{\kappa}} = F(t) \ , \qquad 0 < \kappa < 1$$

is

$$(2.12) \qquad \psi_1(x) = \frac{\sin \kappa\pi}{\pi} \frac{d}{dx} \int_0^x \frac{F(t)dt}{(x-t)^{1-\kappa}} \ .$$

The equation

$$(2.13) \qquad \int_t^1 \frac{\psi_2(x)dx}{(x-t)^{\kappa}} = F(t) \ , \qquad 0 < \kappa < 1$$

can be reduced to (2.11). The substitutions $x = 1 - u$; $t = 1 - v$ change (2.13) into

$$\int_0^v \frac{\psi_2(1-u)du}{(v-u)^{\kappa}} = F(1-v)$$

and then by using (2.12) the solution of (2.13) is

$$(2.14) \qquad \psi_2(x) = -\frac{\sin \kappa\pi}{\pi} \frac{d}{dx} \int_x^1 \frac{F(t)dt}{(t-x)^{1-\kappa}} \ .$$

The application of (2.14) to (2.10) leads to

$$(2.15) \quad x^{1-\gamma}\phi(x) = -\frac{\sin^2\gamma\pi}{\pi^2} \frac{d}{dx} \int_x^1 \frac{1}{(t-x)^{\gamma}} \int_0^t (t-\xi)^{\gamma-1}\xi^{1-\gamma}f(\xi)d\xi dt + \frac{c \sin \gamma\pi}{\pi(1-x)^{\gamma}} \ .$$

Since

$$-\pi \cot \gamma\pi = \lambda$$

and consequently

$$\frac{\sin \gamma\pi}{\pi} = \frac{1}{\sqrt{\lambda^2 + \pi^2}} ;$$

the result (2.15) can also be expressed as

$$(2.16) \quad x^{1-\gamma}\phi(x) = -\frac{1}{\lambda^2+\pi^2}\frac{d}{dx}\int_x^1 \frac{1}{(t-x)^\gamma}\int_0^t (t-\xi)^{\gamma-1}\xi^{1-\gamma}f(\xi)d\xi dt$$

$$+ \frac{c}{(1-x)^\gamma\sqrt{\lambda^2+\pi^2}} .$$

The formula (2.16) is not the formula which is usually given for the solution of (2.5). However, the derivative in (2.16) can be transformed as follows:

$$\frac{d}{dx}\int_x^1 \frac{1}{(t-x)^\gamma}\int_0^t (t-\xi)^{\gamma-1}\xi^{1-\gamma}f(\xi)d\xi dt$$

$$= \frac{d}{dx}\left\{\int_0^x \xi^{1-\gamma}f(\xi)\int_x^1 \frac{(t-\xi)^{\gamma-1}dtd\xi}{(t-x)^\gamma} + \int_x^1 \xi^{1-\gamma}f(\xi)\int_\xi^1 \frac{(t-\xi)^{\gamma-1}dtd\xi}{(t-x)^\gamma}\right\}$$

$$= \frac{d}{dx}\left\{\int_0^x \xi^{1-\gamma}f(\xi)\int_\infty^{\frac{1-\xi}{1-x}} \frac{\sigma^{\gamma-1}d\sigma d\xi}{1-\sigma} + \int_x^1 \xi^{1-\gamma}f(\xi)\int_0^{\frac{1-\xi}{1-x}} \frac{\sigma^{\gamma-1}d\sigma d\xi}{1-\sigma}\right\}$$

$$= x^{1-\gamma} f(x) \int_{\infty}^{1} \frac{\sigma^{\gamma-1} d\sigma}{1-\sigma} - x^{1-\gamma} f(x) \int_{0}^{1} \frac{\sigma^{\gamma-1} d\sigma}{1-\sigma}$$

$$+ \int_{0}^{1} \xi^{1-\gamma} f(\xi) \left(\frac{1-\xi}{1-x}\right)^{\gamma-1} \cdot \frac{1}{1-\left(\frac{1-\xi}{1-x}\right)} \cdot \frac{(1-\xi) d\xi}{(1-x)^2}$$

$$= x^{1-\gamma} f(x) \int_{0}^{\infty} \frac{\sigma^{\gamma-1} d\sigma}{\sigma - 1} + \frac{1}{(1-x)^{\gamma}} \int_{0}^{1} \frac{(1-\xi)^{\gamma} \xi^{1-\gamma} f(\xi) d\xi}{\xi - x}$$

$$= -x^{1-\gamma} f(x)\, \pi \cot \gamma\pi + \frac{1}{(1-x)^{\gamma}} \int_{0}^{1} \frac{(1-\xi)^{\gamma} \xi^{1-\gamma} f(\xi) d\xi}{\xi - x}$$

$$= \lambda x^{1-\gamma} f(x) + \frac{1}{(1-x)^{\gamma}} \int_{0}^{1} \frac{(1-\xi)^{\gamma} \xi^{1-\gamma} f(\xi) d\xi}{\xi - x} \quad .$$

Therefore by using this transformation, which we will denote by T, equation (2.16) can be expressed as

$$(2.17) \quad \phi(x) = - \frac{\lambda f(x)}{\lambda^2 + \pi^2} - \frac{1}{(\lambda^2 + \pi^2)} \cdot \frac{1}{x^{1-\gamma}(1-x)^{\gamma}} \int_{0}^{1} \frac{(1-\xi)^{\gamma} \xi^{1-\gamma} f(\xi) d\xi}{\xi - x}$$
$$+ \frac{c}{x^{1-\gamma}(1-x)^{\gamma} \sqrt{\lambda^2 + \pi^2}}$$

where

$$-\pi \cot \gamma\pi = \lambda$$

The formula (2.17) appears in Muskhelishvili [2] or Mikhlin [3], for instance, and it is regarded as the standard formula for the solution of

$$(2.18) \qquad \int_0^1 \frac{\phi(x)dx}{x - \xi} = \lambda\phi(\xi) + f(\xi) \ .$$

Incidentally, the solution (2.15) can be checked by using successive Abel equations to solve for $f(\xi)$ in (2.15) and then applying a transformation similar to the transformation T noted above.

3. The Cauchy Integral Equation of the Second Kind with a Path of Integration in the Complex Plane

If L is a simple smooth path from α to β in the complex z-plane and if ζ is on L', which denotes L minus its end points, the equation

$$(3.1) \qquad \int_L \frac{\phi(z)dz}{z - \zeta} = \lambda\phi(\zeta) + f(\zeta)$$

is a generalization of (2.18). As is well known (see [2], [3], [4], [5]) the solution of (3.1) can be reduced to the solution of a Hilbert-Riemann boundary value problem. The reduction, as well as the solution of the subsidiary problem, is based on the use of an ingenious idea due to Carleman [6], namely the introduction of the function

$$F(w) = \int_L \frac{\phi(z)dz}{z - w}$$

where w is unrestricted; and the use of the Plemelj formulas. This method for finding the solution of (3.1) is very elegantly effective and it is perhaps indispensable for finding the integral representation for the solution of a Cauchy integral equation of the third kind. For these reasons and others, the method is likely to remain in its position of preeminence. However, apart from an analysis of the properties of $\phi(z)$ which will insure the existence of the Cauchy principal value for all or almost all values of ζ, it could be claimed that the ideas mentioned above require the development of a function theoretic apparatus which is unnecessarily advanced and powerful for the solution of the real equation (2.5) if one grants a knowledge of the integral (2.1). Compared with such a claim, a similar claim with respect to equation (3.1), which involves complex quantities, is less defensible. Nevertheless, the purpose of this section is to show that a solution of (3.1) analogous to that of (2.5) in Section 2 can be synthesized if one is willing to start with the value of a definite integral which is a generalization of (2.1). The generalized integral is

$$(3.2) \qquad -\int_{L_{\alpha\omega}} \left(\frac{\omega-\zeta}{\zeta-\alpha}\right)^{\gamma} \frac{d\zeta}{(\omega-\zeta)(\zeta-z)} , \qquad 0 < \operatorname{Re}(\gamma) < 1$$

where $L_{\alpha\omega}$ is the path along L from α to an arbitrary point ω on L. With the evaluation of this integral as a basis, the solution of (3.1) can be reduced in an elementary way to the solution of a generalization of Abel's equation.

The substitution $\omega - \zeta = \sigma(\zeta - \alpha)$ changes (3.2) into the simpler form

$$(3.3) \qquad \int_{\Gamma_{0\infty}} \frac{\sigma^{\gamma-1} d\sigma}{\sigma(z-\alpha) - (\omega-z)}$$

where $\Gamma_{0\infty}$ denotes a simple smooth path from the origin to infinity in the complex σ-plane. If z is on $L_{\alpha\omega}$, $(\omega-z)/(z-\alpha)$ is on $\Gamma_{0\infty}$; and if z is on $L-L_{\alpha\omega}$, $(\omega-z)/(z-\alpha)$ is not on $\Gamma_{0\infty}$. It is easy to evaluate (3.3) by using the theory of residues and this gives

$$(3.4) \qquad -\int_{L_{\alpha\omega}} \left(\frac{\omega-\zeta}{\zeta-\alpha}\right)^{\gamma} \frac{d\zeta}{(\omega-\zeta)(\zeta-z)}$$

$$= \begin{cases} -\left(\frac{w-z}{z-\alpha}\right)^{\gamma-1} \frac{\pi \cot \gamma\pi}{(z-\alpha)} & z \text{ on } L'_{\alpha\omega} \\ \left(\frac{z-\omega}{z-\alpha}\right)^{\gamma-1} \frac{\pi}{(z-\alpha) \sin \gamma\pi} & z \text{ on } L-L_{\alpha\omega} \end{cases} .$$

The evaluation (3.4) shows that if λ ($\neq \pm\pi i$) is given and γ is chosen so that

$$(3.5) \qquad -\pi \cot \gamma\pi = \lambda , \qquad 0 < \mathrm{Re}(\gamma) < 1 ,$$

then

$$(3.6) \qquad \psi(\zeta,\omega) = \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}}$$

is a solution of the homogeneous equation

$$(3.7) \qquad -\int_{L_{\alpha\omega}} \frac{\psi(\zeta,\omega)d\zeta}{\zeta-z} = \lambda\psi(z,\omega)$$

where z is a point on $L'_{\alpha\omega}$.

Equation (3.1) can be written as

$$(3.8) \qquad \int_{L} \frac{(z-\alpha)\phi(z)dz}{z-\zeta} = \lambda(\zeta-\alpha)\phi(\zeta) + (\zeta-\alpha)f(\zeta) + c$$

where

$$c = \int_{L} \phi(z)dz \ .$$

Multiplication of (3.8) by $(\omega-\zeta)^{\gamma-1}/(\zeta-\alpha)^{\gamma}$ followed by an integration yields

$$(3.9) \qquad \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}} \int_{L} \frac{(z-\alpha)\phi(z)dzd\zeta}{z-\zeta}$$

$$= \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}} [\lambda(\zeta-\alpha)\phi(\zeta) + (\zeta-\alpha)f(\zeta) + c]d\zeta \ .$$

The assumption that the order of integration can be changed in (3.9) leads to

$$(3.10) \qquad -\int_{L_{\alpha\omega}} (z-\alpha)\phi(z) \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}} \frac{d\zeta dz}{\zeta - z}$$

$$-\int_{L_{\omega\beta}} (z-\alpha)\phi(z) \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}} \frac{d\zeta dz}{\zeta - z}$$

$$= \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}}{(\zeta-\alpha)^{\gamma}} [\lambda(\zeta-\alpha)\phi(\zeta) + (\zeta-\alpha)f(\zeta) + c]d\zeta$$

where $L_{\omega\beta}$ is the path along L from ω to β. If γ is chosen so that (3.5) holds then (3.4) shows that (3.10) is the same as

$$(3.11) \qquad \frac{\pi}{\sin \gamma\pi} \int_{L_{\omega\beta}} \frac{(z-\alpha)^{1-\gamma}}{(z-\omega)^{1-\gamma}} \phi(z)dz$$

$$= \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma}f(\zeta)d\zeta}{(\omega-\zeta)^{1-\gamma}} + c \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}d\zeta}{(\zeta-\alpha)^{\gamma}} \quad .$$

Since

$$(3.12) \qquad \int_{L_{\alpha\omega}} \frac{(\omega-\zeta)^{\gamma-1}d\zeta}{(\zeta-\alpha)^{\gamma}} = \frac{\pi}{\sin \gamma\pi}$$

equation (3.11) reduces to

$$(3.13) \qquad \int_{L_{\omega\beta}} \frac{(z-\alpha)^{1-\gamma}\phi(z)dz}{(z-\omega)^{1-\gamma}} = \frac{\sin \gamma\pi}{\pi} \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma}f(\zeta)d\zeta}{(\omega-\zeta)^{1-\gamma}} + c$$

which may be classified as a generalized Abel equation. The equation can be solved by using (2.14). However, for completeness, a derivation of the solution follows.

Equation (3.13) can be solved by multiplying each side by $1/(\omega-\tau)^{\gamma}$ where τ is on $L_{\alpha\beta}$; and integrating along L from τ to β. This gives

$$\int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\omega\beta}} \frac{(z-\alpha)^{1-\gamma}\phi(z)dzd\omega}{(z-\omega)^{1-\gamma}}$$

$$= \int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\tau\beta}} \frac{(z-\alpha)^{1-\gamma}\mu(z-\omega)\phi(z)dzd\omega}{(z-\omega)^{1-\gamma}}$$

where

$$\mu(z-\omega) = \begin{cases} 1 & \text{if } z \text{ follows } \omega \text{ along } L \\ 0 & \text{if } z \text{ precedes } \omega \text{ along } L\,. \end{cases}$$

Hence it can be seen that

$$\int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\omega\beta}} \frac{(z-\alpha)^{1-\gamma}\phi(z)dzd\omega}{(z-\omega)^{1-\gamma}}$$

$$= \int_{L_{\tau\beta}} (z-\alpha)^{1-\gamma}\phi(z) \int_{L_{\tau\beta}} \frac{\mu(z-\omega)d\omega dz}{(\omega-\tau)^{\gamma}(z-\omega)^{1-\gamma}}$$

$$= \int_{L_{\tau\beta}} (z-\alpha)^{1-\gamma}\phi(z) \int_{L_{\tau z}} \frac{d\omega dz}{(\omega-\tau)^{\gamma}(z-\omega)^{1-\gamma}}$$

or by using (3.12)

$$\int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\omega\beta}} \frac{(z-\alpha)^{1-\gamma}\phi(z)\,dz\,d\omega}{(z-\omega)^{1-\gamma}} = \frac{\pi}{\sin\gamma\pi} \int_{L_{\tau\beta}} (z-\alpha)^{1-\gamma}\phi(z)\,dz .$$

From this, the solution of (3.13) is given by

$$(3.14) \qquad \frac{\pi}{\sin\gamma\pi} \int_{L_{\tau\beta}} (z-\alpha)^{1-\gamma}\phi(z)\,dz$$

$$= \frac{\sin\gamma\pi}{\pi} \int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta\,d\omega}{(\omega-\zeta)^{1-\gamma}} + c \int_{L_{\tau\beta}} \frac{d\omega}{(\omega-\tau)^{\gamma}} .$$

The derivative of (3.14) is

$$(3.15) \qquad (\tau-\alpha)^{1-\gamma}\phi(\tau)$$

$$= -\frac{\sin^2\gamma\pi}{\pi^2} \frac{d}{d\tau} \int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta\,d\omega}{(\omega-\zeta)^{1-\gamma}} + \frac{c\,\sin\gamma\pi}{\pi(\beta-\tau)^{\gamma}}$$

which demonstrates that the formula for the solution of (3.13) is analogous to the formula (2.15) for the solution of (2.10). Since $-\pi\cot\gamma\pi = \lambda$, the result (3.15) can also be written as

$$(3.16) \quad (\tau-\alpha)^{1-\gamma}\phi(\tau) = -\frac{1}{\lambda^2+\pi^2} \frac{d}{d\tau} \int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta\,d\omega}{(\omega-\zeta)^{1-\gamma}}$$

$$+ \frac{c}{(\beta-\tau)^{\gamma}\sqrt{\lambda^2+\pi^2}} .$$

The derivative of the double integral in (3.16) can be replaced by

$$\frac{d}{d\tau}\int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\alpha\omega}} \frac{(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta d\omega}{(\omega-\zeta)^{1-\gamma}}$$

$$= \frac{d}{d\tau}\left\{\int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\alpha\tau}} \frac{(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta d\omega}{(\omega-\zeta)^{1-\gamma}}\right.$$

$$\left. + \int_{L_{\tau\beta}} \frac{1}{(\omega-\tau)^{\gamma}} \int_{L_{\tau\beta}} \frac{\mu(\omega-\zeta)\cdot(\zeta-\alpha)^{1-\gamma} f(\zeta)\,d\zeta d\omega}{(\omega-\zeta)^{1-\gamma}}\right\}$$

$$= \frac{d}{d\tau}\left\{\int_{L_{\alpha\tau}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{L_{\tau\beta}} \frac{(\omega-\zeta)^{\gamma-1} d\omega d\zeta}{(\omega-\tau)^{\gamma}}\right.$$

$$\left. + \int_{L_{\tau\beta}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{L_{\tau\beta}} \frac{(\omega-\zeta)^{\gamma-1}}{(\omega-\tau)^{\gamma}} \cdot \mu(\omega-\zeta)\,d\omega d\zeta\right\}$$

$$= \frac{d}{d\tau}\left\{\int_{L_{\alpha\tau}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{L_{\tau\beta}} \frac{(\omega-\zeta)^{\gamma-1} d\omega d\zeta}{(\omega-\tau)^{\gamma}}\right.$$

$$\left. + \int_{L_{\tau\beta}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{L_{\zeta\beta}} \frac{(\omega-\zeta)^{\gamma-1} d\omega d\zeta}{(\omega-\tau)^{\gamma}}\right\}$$

$$= \frac{d}{d\tau}\left\{\int_{L_{\alpha\tau}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{\infty}^{\frac{\beta-\zeta}{\beta-\tau}} \frac{\sigma^{\gamma-1} d\sigma d\zeta}{1-\sigma}\right.$$

$$\left. + \int_{L_{\tau\beta}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \int_{0}^{\frac{\beta-\zeta}{\beta-\tau}} \frac{\sigma^{\gamma-1} d\sigma d\zeta}{1-\sigma}\right\}$$

$$= +(\tau-\alpha)^{1-\gamma} f(\tau) \int_0^\infty \frac{\sigma^{\gamma-1} d\sigma}{\sigma - 1}$$

$$+ \int_{L_{\alpha\beta}} (\zeta-\alpha)^{1-\gamma} f(\zeta) \left(\frac{\beta-\zeta}{\beta-\tau}\right)^{\gamma-1} \frac{1}{1 - \left(\frac{\beta-\zeta}{\beta-\tau}\right)} \cdot \frac{(\beta-\zeta)}{(\beta-\tau)^2} \cdot d\zeta$$

$$= \lambda(\tau-\alpha)^{1-\gamma} f(\tau) + \frac{1}{(\beta-\tau)^\gamma} \int_{L_{\alpha\beta}} \frac{(\zeta-\alpha)^{1-\gamma}(\beta-\zeta)^\gamma f(\zeta) d\zeta}{\zeta - \tau} \quad .$$

Hence (3.16) can be expressed as

$$(3.17) \quad \phi(\tau) = -\frac{\lambda f(\tau)}{\lambda^2+\pi^2} - \frac{1}{(\lambda^2+\pi^2)(\tau-\alpha)^{1-\gamma}(\beta-\tau)^\gamma} \int_{L_{\alpha\beta}} \frac{(\zeta-\alpha)^{1-\gamma}(\beta-\zeta)^\gamma f(\zeta) d\zeta}{\zeta - \tau}$$

$$+ \frac{c}{(\tau-\alpha)^{1-\gamma}(\beta-\tau)^\gamma \sqrt{\lambda^2 + \pi^2}}$$

where

$$-\pi \cot \gamma\pi = \lambda \ .$$

This is the form which is usually presented for the solution of

$$(3.18) \qquad \int_{L_{\alpha\beta}} \frac{\phi(z) dz}{z - \zeta} = \lambda\phi(\zeta) + f(\zeta) \ .$$

It is to be noticed that although (3.17) depends only on a single integral the integrand contains the factor $1/(\zeta-\tau)$ and hence the integral denotes the Cauchy principal value. This factor does not appear in (3.16). For some purposes the formula (3.16) may be more advantageous than (3.17).

It is interesting to note that the foregoing analysis aubsumes the solution of

$$(3.19) \qquad \oint_C \frac{\phi(z)dz}{z - \zeta} = \lambda\phi(\zeta) + f(\zeta)$$

where C is a simple smooth closed path and ζ is on C. If α is any point on C it can be taken as both the initial and end point of a path $L_{\alpha\beta}$ which coincides with C. Hence the solution of (3.19) can be found by setting $\beta = \alpha$ in (3.16) or (3.17). The latter form gives

$$(3.20) \qquad \phi(\tau) = -\frac{\lambda f(\tau)}{\lambda^2 + \pi^2} - \frac{1}{(\lambda^2 + \pi^2)(\tau - \alpha)} \oint_C \frac{(\zeta - \alpha) f(\zeta) d\zeta}{\zeta - \tau}$$

$$+ \frac{c}{(\tau-\alpha)^{1-\gamma}(\alpha-\tau)^{\gamma}\sqrt{\lambda^2 + \pi^2}}$$

or

$$(3.21) \qquad \phi(\tau) = -\frac{f(\tau)}{\lambda^2 + \pi^2} - \frac{1}{(\lambda^2 + \pi^2)} \oint_C \frac{f(\zeta)d\zeta}{\zeta - \tau}$$

$$- \frac{1}{(\lambda^2 + \pi^2)(\tau - \alpha)} \left[\oint_C f(\zeta)d\zeta - \frac{\pi e^{-\pi i \gamma}}{\sin \gamma\pi} \oint_C \phi(\zeta)d\zeta \right] .$$

A solution of (3.19) must satisfy

$$\oint_C \oint_C \frac{\phi(z)dz d\zeta}{z - \zeta} = \lambda \oint_C \phi(\zeta)d\zeta + \oint_C f(\zeta)d\zeta$$

or

$$-\pi i \oint_C \phi(z)dz = \lambda \oint_C \phi(\zeta)d\zeta + \oint_C f(\zeta)d\zeta .$$

That is,

$$\oint_C f(\zeta)d\zeta = -(\lambda + \pi i) \oint_C \phi(\zeta)d\zeta$$

and since $-\pi \cot \gamma\pi = \lambda$,

$$\oint_C f(\zeta) = \pi(\cot \gamma\pi - i) \oint_C \phi(\zeta)d\zeta$$

$$= \frac{\pi e^{-i\gamma\pi}}{\sin \gamma\pi} \cdot \oint_C \phi(\zeta)d\zeta$$

which shows that the quantity within the bracket of (3.21) is zero. Therefore the solution of

$$\oint_C \frac{\phi(z)dz}{z - \zeta} = \lambda\phi(\zeta) + f(\zeta)$$

is

$$\phi(\tau) = -\frac{\lambda f(\tau)}{\lambda^2 + \pi^2} - \frac{1}{(\lambda^2 + \pi^2)} \oint_C \frac{f(\zeta)d\zeta}{\zeta - \tau} .$$

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